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The General Semimartingale Model with Dividends

Abstract

This paper delves into the evolution and intricacies of financial mathematics, tracing its roots from Bachelier's groundbreaking work in 1900 to the comprehensive financial market models of the 1990s. While the general market model postulated by Delbaen in 1998 serves as an inclusive framework, certain gaps and limitations persist, particularly concerning non-discounted setups, potentially negative price processes, and dividend considerations. The aim of the contribution is to bridge these gaps by presenting a general market model that encompasses dividend payments in real-world contexts, transitioning subsequently to a discounted setup. We define such a market and find the necessary technical requirements.

Key words

Stochastic Analysis, Financial Mathematics, General Market Model, Dividend Modelling

JEL Classification

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Introduction

The theory of modern financial mathematics in its present form has its origin in the dissertation "Théorie de la Spéculation" by L.F. Bachelier from 1900 (Bachelier, 1900). In this publication, one finds the first mathematical description of Brownian motion as a stochastic process (although not under that name). Bachelier's goal was to derive theoretical values for various types of options on certain goods by modelling prices of goods using a Brownian motion and comparing these prices with actual market prices. He proposed as option prices the expected value of the payment arising from the option. The crucial shortcoming in Bachelier's modelling was that the prices of goods could become negative.

Bachelier's work was forgotten for a long time. It was only after the development of the stochastic integral and the introduction of geometric Brownian motion as a pricing model in the 1960s that financial mathematics revived (Itô, 1944; Itô, 1946; Itô, 1950; Itô, 1951a; Itô, 1951b; Itô, 1951c).

In 1973, Fischer Black and Myron Scholes made the decisive breakthrough (Scholes et al., 1973) by developing the famous Black-Scholes equation and formula.

Since then, financial mathematics has become a huge field of research, and numerous models have been proposed and analysed. The progress and advancement of stochastic analysis and stochastic integral, mainly by Doob (1953), Meyer (1962), Meyer (1963), Kunita et al. (1967), Meyer (1967a), Meyer (1967b), Meyer (1967c), Meyer (1967d), Doléans-Dade et al. (1970), Meyer (2002), Jacod (1979), Chou et al. (1980), and Jacod (1980) also opened up numerous new possibilities for financial mathematics. In particular, the modern approach of option pricing according to the duplication principle has established itself as a standard. This approach is a natural application of the martingale theory and representation theorem. Here Harrison et al. (1981) can be considered as a cornerstone.

However, at that time, while many different models had been examined and studied, there was no overarching theory that combined all these models to lay the groundwork for modelling financial markets. A significant breakthrough in the general theory of financial mathematics was achieved in the 1990s by Delbaen et al. (1994) and Delbaen et al. (1998) by presenting a very general financial market model that included almost all of the known models and proving the connection between arbitrage and mathematical conditions on the existence of specific probability measures. Since then, most publications have referred to this model. The financial market, as discussed in Delbaen et al. (1998), can be seen as the general market, which comprises almost all models of frictionless markets that are used in practice. Therefore, the results are universal, and the general set-up of the market is, without a doubt, the most important market model in Mathematical Finance. However, there are some gaps and shortcomings in the literature.

- The model in Delbaen et al. (1998) assumes a discounted set up (sometimes also referred as normalized set up). However, most models used in practice are described in non-discounted terms in order to be able to verify its assumption with real-world observations. The question how and under which assumptions non-discounted set ups can be transformed to discounted set-ups has not been described in general terms, but only for specific models.
- The price processes in Delbaen et al. (1998) are potentially negative. The price processes of any most results of the general theory, such as the Second Fundamental

Theorem of Asset Prices, the Third Fundamental Theorem of Asset Prices, or the theory of bubbles are assumed to be bounded from below. A generalisation to the original set-up has not taken place yet.

- The model as it is presented in Delbaen et al. (1998) does not consider dividends or additional cash-flows, and therefore excludes some essential models, such as models for pricing and setting of futures. In many cases, it is possible to transform dividend-paying models to non-dividend models (see for example Jarrow, 2021, Section 2.3). Therefore, an extension of the initial model to include dividends is desirable.
- Some basic properties of market models are often assumed to be true without validating them for this very general model. This applies to the notion of admissible strategies, discounted processes, numéraires and so on. In particular, since the general market model allows for negative prices some of the available properties get more complicated or even completely devalidated.

The aim of this contribution is to close the gaps mentioned above and to define a general market model that considers dividend payments in a real-world set-up. Then introduce the notion of a numéraire and transform the model to a discounted set-up. The special conditions and technical requirements are investigated and motivated. Furthermore, the literature on these topics and some examples are reviewed.

1 The General Semimartingale Model with Dividends

In this section, we are going to define a very general market model with dividends.

Our discussion primarily centers on time-continuous models. Although discrete models can be viewed as a subset of time-continuous models, they are typically easier to navigate mathematically. Nevertheless, time-continuous models are arguably more popular in financial mathematics, especially within portfolio theory. A key rationale for not exclusively relying on discrete models is that optimization problems in such a domain ideally have unique solutions. However, this uniqueness is often absent in discrete-time strategies, leading practitioners to settle for an approximate series of trading strategies.

For a given $\mathbb{R}^d$-valued semimartingale S, the space $L(S)$ is defined as the set of possible integrands for S for the general vector-valued stochastic integrals for a semimartingale integrator. Furthermore for a semimartingale $S, \varphi \in L(S)$ and $t \in \mathbb{R}_+$

$$\varphi_0^\top S_0 + \int_{(0,t]} \varphi \mathrm{d} S_u$$

denotes the stochastic integral at time t.

Consider a financial market that comprises $d+1$ tradable securities. The price processes of these securities are depicted by the $d+1$-dimensional process $S_t = \left(S_t^0, S_t^1, \dots, S_t^d\right)_{t\in\mathbb{R}_+}$. These processes also have associated cumulative dividend processes, termed $D = \left(D_t^0, D_t^1, \dots, D_t^d\right)_{t\in\mathbb{R}_+}$, which are adapted to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$

It is assumed that the market is both frictionless and competitive. A market that is referred to as "frictionless" is characterized by the absence of transaction costs, differential taxes, and trading constraints, including short sale restrictions, borrowing limits, and margin requirements. Furthermore, within this particular market, it is worth noting that shares possess the characteristic of being infinitely divisible. The term "competitive" in reference to the market denotes a situation where traders function as price takers. In this context, individuals have the ability to engage in trading activities involving any desired quantity of shares without exerting any influence on the market price. This ensures that there is no presence of liquidity risk.

Furthermore, the following mathematical assumptions are being made:

- $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ is a filtered probability space with probability measure $\mathbf{P}$.
- The processes S_t^i and D_t^i are semimartingales for all $i = 0, \dots, d$. [1]
- The filtration $\mathbb{F}$ satisfies the usual conditions and the σ-algebra $\mathcal{F}_0$ is trivial, that is, $A \in \mathcal{F}_0$ implies $\mathbf{P}(A) = 0$ or $\mathbf{P}(A) = 1$. [2]

Remark 1.1. Markets with dividends can often be transformed into dividend-free ones. However, these transformations usually need additional assumptions to hold that are not universally applicable. For instance, Jarrow, 2021 presumes all cash flows to be positive and that the numéraire follows a deterministic continuous FV process.

The process D_t^i denotes the cumulative dividend payments of the i-th share up until the time t. We don't mandate monotonicity; hence, distributions might even be negative (additional distribution). Dividend processes that potentially have negative increments are crucial

[1] There are several definitions for semimartingales in the literature, see for example Cohen et al. (2015) vs. Protter (2010). However, they are equivalent, as it is shown in Section 3.9 in Protter (2010).

[2] The usual conditions are, for example, defined in Cohen et al. (2015) p. 139 or Protter (2010) p. 3 as usual hypotheses.

for the study of certain products such as futures. Such dividend payments can be viewed as cash flows, making the limitation to solely positive cash flows appear overly stringent.

Generally, dividend processes are categorized into two types:

The first type posits continuous paths for the dividend processes, which, while mathematically convenient, isn't particularly realistic. The second assumes discrete dividend payments, with the dividend process D being a pure jump process[3] $(D = \sum \Delta D)$. This is more aligned with reality, though not straightforward mathematically.

Mathematically speaking, it's prudent to presume price processes as semimartingales because the most widely accepted definition of the general stochastic integral only incorporates semimartingales as integrators. This assumption also finds economic justification; for instance, if the price process is locally bounded, adapted, and the market adheres to the 'No Free Lunch With Vanishing Risk' principle (a variant of 'No Arbitrage'), then the price process is a semimartingale, as demonstrated in Theorem 8 of Ansel et al. (1992) and on pages 504-507 of Delbaen et al. (1994). Another economic rationale is provided in Kardaras et al. (2011), where Constantinos Kardaras and Eckhard Platen show that in markets where only simple predictable trading strategies are permitted, where short-selling is disallowed and no-arbitrage principles hold, price processes are always semimartingales[4].

Requiring all price processes to be semimartingales excludes fractional Brownian motions with a Hurst parameter $H \neq \frac{1}{2}$. As of now, a consistent no-arbitrage theory for these processes in a frictionless, continuously trading market is non-existent. Yet, in markets with transaction costs, such arbitrage possibilities typically vanish. In these settings fractional Brownian motions are considered realistic and reasonable, for example, in Guasoni (2006).

1.1 Self-financing Trading Strategies

Definition 1.2. (a) *A $d+1$-dimensional process $\varphi = (\varphi^0, \dots, \varphi^{d+1}) \in L(S)$ is called a trading strategy.*

(b) The wealth process of the investor is defined as

$$V_t := \sum_{i=0}^{d} \varphi_t^i \left(S_t^i + \Delta D_t^i\right),\ t \in \mathbb{R}_+$$

where φ_t^i represents the number of the i-th security that an investor holds in his portfolio at t.

[3] For the pure jump process definition, refer to Klebaner (2005, Chapter 9).
[4] However, the No Arbitrage definition in this paper deviates from NFLVR.

Definition 1.3. *A trading strategy* $\varphi = (\varphi^0, \varphi^1, \dots, \varphi^d)$ *is called self-financing if the wealth process* $V_t(\varphi)$ *satisfies*

$$V_t = \int_0^t \varphi_s \, d(S_s + D_s), \ \textit{for all } t \in \mathbb{R}_+$$

Note the equation deals with higher dimensional processes and hence multi-dimensional integration.

Remark 1.5. The variable φ_t^i represents the quantity of securities i held by the agent within the portfolio, specifically within the time frame of t - to t (referred to as the investment in ΔS_t^i). The jumps referred to as ΔS_t^i and ΔD_t^i should be considered as synchronous. The term S^i can be conceptualised as the ex-dividend price, which refers to the price of a security after the dividend payment has been made.

The dividend payout for the agent at time t is $\sum_{i=0}^{d} \varphi_t^i \Delta D_t^i$. This is now invested in the $d+1$ securities immediately after t. So, as before, there is no permanent cash holding. Therefore, only ΔD and not D appears in 1.

1.2 Numéraire

In practice, comparing two assets at different times based on their nominal size is unusual and makes no sense. Therefore, a benchmark should be introduced that enables us to produce comparative values. A numéraire represents this benchmark.

Definition 1.6. *A predictable semimartingale* $(N_t)_{t\in\mathbb{R}_+}$ *which satisfies* $\inf_{t\in[0,T]} N_t > 0$, *for all* $T \in \mathbb{R}_+ \boldsymbol{P} -$ *a.s. is called a numéraire.*

Remark 1.7. The predictability is necessary to ensure that $\frac{1}{N}$ can be used as integrand. This is important to define the discounted dividend process and to map the self-financing condition of price processes to the self-financing condition of their discounted counterparts.

If one writes asset values as multiples of the numéraire, they can also be compared independently of the time. The discounted processes are denoted by $\hat{S}^i$ or $\hat{V}$, which means:

$$\hat{S}^i := \frac{S^i}{N} \text{ and } \hat{V} := \frac{V}{N}$$

For further analysis, one needs the following result.

Lemma 1.8. *Let* N *be a numéraire, then* $\frac{1}{N}$ *is bounded on each compact interval and in particular locally bounded.*

Proof. Given the definition of a numéraire, for every $T \in \mathbb{R}_+$, with probability 1, the infimum of N_t over the interval $[0, T]$ is strictly positive. This implies that for any compact interval $[a, b] \subset \mathbb{R}_+$, there exists a $\delta > 0$ such that $N_t > \delta$ for all $t \in [a, b]$ almost surely.

Considering the reciprocal function $\frac{1}{N_t}$, since $N_t > \delta$ for all $t \in [a, b]$ almost surely, we deduce $\frac{1}{N_t} < \frac{1}{\delta}$ for all $t \in [a, b]$ almost surely. Thus, the function $\frac{1}{N_t}$ is bounded by $\frac{1}{\delta}$ on the compact interval $[a, b]$ almost surely. Since this argument holds for any compact interval $[a, b] \subset \mathbb{R}_+$, we conclude that $\frac{1}{N}$ is locally bounded. □

The literature does not present a unique definition for the numéraire. For instance, in works like Bingham et al. (2013), Elliott et al. (2005), or Pascucci (2011), it is only required that N should be a positive semimartingale. This notion also extends to Geman et al. (1995), regarded as the standard reference for a numéraire in an abstract context. However, to ensure that discounted price processes remain semimartingales, the numéraire's left limit must also be greater than 0. Even stricter conditions are demanded in specific settings such as Ebenfeld (2007). Additionally, distinctions between strong and weak numéraire concepts are discussed in (Klein et al., 2016). Our interpretation aligns with the one presented in Qin et al. (2017). Recent publications have started to describe financial market models without invoking the concept of a numéraire. Nevertheless, more intricate market assumptions, as described in (Herdegen et al., 2016), are necessary to maintain the feasibility of discounting price processes.

It is essential to introduce both a discounted dividend process, $\hat{D}^i$, and a discounted wealth process, $\hat{V}$. The process $\hat{D}_t$ denotes cumulative dividends up to a given time. However, each dividend gets discounted by the numéraire value, N, at its payment time - not by its value at t. This approach ensures that any change in the discounted dividend process, $\hat{D}^i$, occurs only when the dividend process, D^i, changes. In scenarios like $\frac{D^i}{N}$, these properties would not hold since a mere change in the numéraire N could lead to a change in $\frac{D^i}{N}$. Therefore, the following definition naturally arises.

Definition 1.9. *By* $\hat{D}^i_t := \int_0^t \frac{1}{N}\, dD^i_u, i = 0, \dots, d$ *the discounted dividend processes is denoted. Furthermore*

$$\hat{V}(\varphi) := \frac{V}{N}(\varphi) = \sum_{i=0}^{d} \varphi^i\left(\hat{S}^i + \Delta\hat{D}^i\right)$$

denotes the discounted wealth process.

Remark 1.10. By Protter (2010) Theorem IV.18, one has $\Delta\hat{D}^i = \frac{\Delta D^i}{N}$.

While it is commonly assumed that D is a pure jump process, we want to avoid making this assumption here and rather stay as general as possible.

However, if we operate under the assumption that the dividend process is a pure jump process (meaning $D = \sum \Delta D$ is valid), then, by Theorem IV.17 and Theorem IV.18 from Protter (2010), we arrive at

$$\hat{D} = \int_0^t \frac{1}{N_s} \,\mathrm{d}D_s^i = \int_0^t \frac{1}{N_s} \,\mathrm{d}\left(\sum_{0\leq u\leq s} \Delta D_u^i\right) = \sum_{0\leq s\leq t} \Delta\left(\int_0^s \frac{1}{N_u} \,\mathrm{d}D_u^i\right) = \sum_{0\leq s\leq t} \Delta\hat{D}_s^i.$$

In this scenario, $\hat{D}$ also becomes a pure jump process, only changing when D does.

The following theorem is immensely beneficial, offering the foundational technical knowledge to explore properties in discounted settings using the conventional tools of stochastic analysis.

Theorem 1.11. *The processes $N, S^0, \dots, S^d, D^0, \dots, D^d$ are semimartingales and N is a numéraire if and only if the processes $\frac{1}{N}, \hat{S}^0, \dots \hat{S}^d, \hat{D}^0, \dots, \hat{D}^d$ are semimartingales and $\frac{1}{N}$ is a numéraire. If, furthermore, φ is self-financing, then the discounted wealth process $\hat{V}(\varphi)$ is a semimartingale.*

Proof. $\Rightarrow$ Let N be a numéraire and $S^0, \dots, S^d$ semimartingales. We define

$$T^n := \inf\left\{t \geq 0; N_t \leq \frac{1}{n}\right\}$$

Now (T^n) is a localizing sequence and we examine the processes

$$N_t^n := (N^n)_t^{T^n-} := \begin{cases} N_t & \text{for } t < T^n \\ N_{T_n-} & \text{for } t \geq T^n \end{cases}$$

Let f_n be a convex function, which satisfies $f_n(x) = \frac{1}{x}$ for $x \geq \frac{1}{n}$. By Remark 3.2, we obtain that $f_n(N^n)$ is a semimartingale and since $N^n \geq \frac{1}{n}$ holds, $\frac{1}{N^n}$ is also a semimartingale. Thus $\frac{1}{N^n}$ is prelocally a semimartingale and hence, by Theorem 3.3, a semimartingale. Since N is a semimartingale and therefore in particular càdlàg, we also have

$$\mathbf{P}\left(\sup_{t\in[0,T]} N_t < \infty\right) = 1 \ \text{ for all } T \in \mathbb{R}_+$$

and hence

$$\inf_{t\in[0,T]} \frac{1}{N_t} > 0 \text{ for all } T \in \mathbb{R}_+ \ \mathbf{P}- \text{ almost surely.}$$

Thus $\frac{1}{N}$ is a numéraire and $\hat{S}^i = \frac{S^i}{N}$ is by Theorem 3.1 also a semimartingale.

$\Leftarrow$ Let $\frac{1}{N}, \hat{S}^0, \dots, S^d$ be semimartingales and $\frac{1}{N}$ a numéraire. If we proceed as above, we obtain that $N = \frac{1}{\frac{1}{N}}$ is a semimartingale and by arguing as above, it follows that N is càdlàg. Hence N is a numéraire. Since

$$\hat{\hat{S}}^i = \frac{\hat{S}^i}{\frac{1}{N}} = S^i \ i = 0, \dots d$$

holds, we obtain that S^i are also semimartingales.

Now we assume φ to be self-financing. Then we have $\hat{V}_t = \frac{\int_0^t \varphi \mathrm{d}S_u}{N_t}$, which is a semimartingale by Theorem 3.1.

The subsequent theorem asserts that the self-financing property stays valid regardless of whether the associated wealth process is viewed in nominal or discounted terms.

Theorem 1.12. *Let $\varphi = (\varphi^0, \varphi^1, \dots, \varphi^d)$ be a trading strategy, V be the associated wealth process and N a numéraire with $[N, D^i]^c = 0$ for all $i \in \{0, \dots, d\}$. Then φ is self-financing if and only if*

$$\hat{V}_t(\varphi) = \sum_{i=0}^{d} \int_0^t \varphi_u^i \, d\left(\hat{S}_u^i + \hat{D}_u^i\right), \ t \in [0, T]$$

where $\hat{v}_0 := \frac{v_0}{N_0}$

Proof. We show the one-dimensional case. The multi-dimensional case follows from the linearity of the integral and the fact that any strategy can be approximated by component-wise integrable strategy.

By the definition of V and $\hat{V}$, it is easy to see that $V_- = \varphi S_-$ and $\hat{V}_- = \varphi \hat{S}_-$.

We first assume

$$\hat{V}_t(\varphi) = \int_0^t \varphi_u \, \mathrm{d}\left(\hat{S}_u + \hat{D}_u\right).$$

By Theorem 1.11, $\hat{V} = \hat{V}(\varphi)$ is a semimartingale and with Theorem 3.4, we obtain with a slight abuse of notation

$$\begin{aligned}
V_t = \hat{V}_t N_t &= \int_0^t N_{s-} \mathrm{d}\hat{V}_s + \int_0^t \hat{V}_{s-} \mathrm{d}N_s + [N, \hat{V}]_t \\
&= \int_0^t N_{s-} \mathrm{d}\left(\int_0^s \varphi_u \, \mathrm{d}(\hat{S} + \hat{D})_u\right) + \int_0^t \left(\varphi \hat{S}_{u-}\right) \mathrm{d}N_u + \left[N_t, \int_0^t \varphi \mathrm{d}\left(\hat{S}_u + \hat{D}_u\right)\right] \\
&= \int_0^t \varphi \mathrm{d}\left(\int_0^s N_{u-} \mathrm{d}\hat{S}_u + \int_0^s (N_u - \Delta N_u) \mathrm{d}\hat{D}_u + \int_0^s \hat{S}_{u-} \mathrm{d}N_u + \left[N_s, \hat{S}_s\right] + \left[N_s, \hat{D}_s\right]\right) \\
&= \int_0^t \varphi \mathrm{d}\left(N_s \hat{S}_s + \int_0^s N_u \, \mathrm{d}\hat{D}_u - \sum_{0 \le u \le s} \Delta N_u \Delta \hat{D}_u + [N, \hat{D}]_s^c + \sum_{0 \le u \le s} \Delta N_u \Delta \hat{D}_u\right) \\
&= \int_0^t \varphi_s \, \mathrm{d}\left(N_s \hat{S}_s + \int_0^s N_u \, \mathrm{d}\left(\int_0^u \frac{1}{N_r} \, \mathrm{d}D_r\right)\right) \\
&= \int_0^t \varphi_s \, \mathrm{d}(S_s + D_s)
\end{aligned}$$

Hence φ is self-financing. The converse follows analogously with $N' = \frac{1}{N}$ and $\hat{V} = VN'$.

Since each trading strategy can be approximated with component-wise integrable trading strategy, the general result follows by taking limits. □

2 Conclusion

This paper explores the evolution of financial mathematics, starting with Bachelier's early work in 1900 and ending with the comprehensive market models of the 1990s. Despite its breadth, Delbaen's 1998 model had limitations, such as not accounting for non-discounted settings, the possibility of negative price paths, and importantly, dividends.

To address these issues, we developed a general model including dividends. We carefully defined each component. When we moved from a standard to a discounted model, we made sure to close literature gaps and found that traditional features often changed or became irrelevant, particularly in dividend models. However, we pinpointed necessary technical requirements to keep these features consistent. Key among these are the predictability and lower bound limits of the numeraire, as well as the uncorrelatedness of the continuous part of dividends and numeraire. By combining a thorough review of relevant literature with our innovative insights, this paper manages to fill existing gaps and strengthen the foundations of financial mathematics.

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Appendix:

Some referenced results

A proof for the following result can be found in Protter, 2010, Corollary 2 of Theorem 2.22.

Theorem 3.1 (Partial integration). Let X, Y be two semimartingales. Then XY is a semimartingale and

$$X_t Y_t = \int_0^t X_{s-} \mathrm{d}Y_s + \int_0^t Y_{s-} \mathrm{d}X_s + [X, Y]_t$$

Remark 3.2. The space of semimartingales posseses some remarkable stability properties. For example, for a convex function $f: \mathbb{R} \to \mathbb{R}$ and a semiartingale $X, f(X)$ is also semimartingale. This is a consequence of the Tanaka-Meyer-Itô Rule (see, for example, Cohen et al. (2015) Theorem 14.3.11 or Protter (2010) Theorem IV.70). Theorem IV.66 from Protter (2010) also provides a simplified proof of the abovementioned fact.

The next result is taken from Protter, 2010, Chapter II Section 2

Theorem 3.3. (a) Local semimartingales and processes that are prelocally semimartingales are semimartingales.

(b) An $\mathbb{R}^d$-valued stochastic process X is a d-dimensional semimartingale if and only if all components are one-dimensional semimartingales.

(c) The set of all semimartingales form a vector space.

(d) Let **Q** be a probability measure that is absolutely continuous with respect to **P**. Then every **P**-semimartingale is also a **Q**-semimartingale.

The following result can be found in Protter, 2010, Chapter II Section 6

Theorem 3.4. Let $X, Y \in \mathcal{S}^1$. The process $[X, Y]$ is an FV process, a semimartingale and has the following properties.

(a) $[X, Y]_0 = X_0 Y_0$ and $\Delta[X, Y] = \Delta X \Delta Y$.

(b) Let T be a stopping time. Then we have

$$[X^T, Y] = [X, Y^T] = [X^T, Y^T] = [X, Y]^T$$

(c) The quadratic variation $[X, X]$ is a positive, increasing process.

(d) If X is an FV process, we have

$$[X,Y]_t = X_0 Y_0 + \sum_{0<s\leq t} \Delta X_s \Delta Y_s$$

(e) For $X, Y \in \mathcal{S}^1, H \in L(X)$ and $K \in L(Y)$, we have

$$[H \cdot X, K \cdot Y]_t = \int_0^t H_s K_s \, \mathrm{d}[X,Y]_s \ (t \geq 0)$$