Moritz Sohns

Utility indifference pricing in the Heston model: pricing, hedging and shortcomings

Abstract

This paper deals with the Heston model's utility indifference pricing method via the exponential utility function. We illustrate the main properties, review the existing literature and elaborate on the idea behind the pricing method and control. The main results of this paper are a pricing equation for the model, an equation for the optimal hedging strategy in the model, an illustration of why short positions must not appear in practice when applying the utility indifference approach, which is a contradiction to the observed real-world trading and a simulation of the price process, calculation of the corresponding derivative prices and a comparison of the different hedging strategies.

The simulation together with the result about the absence of short positions hints that utility indifference pricing should be treated with caution when applied in practice in the real world.

Keywords

Heston model, stochastic volatility, derivatives pricing, option pricing, pricing simulation

JEL classification

C51, C52, G12, G13, G17

Introduction

The Black-Scholes model has become the standard model for valuing derivatives in industry and academia. The distribution of neither the underlying price process of the stock nor the prices for underlying match the empirical distribution of assets traded in the real world (Rubinstein, 1994; Duan, 1999). One particular drawback of the Black-Scholes model seems to be the assumption of stationary volatility Black; Scholes, 1973 and data shows that the volatility is random (Blattberg et al., 1974, Scott, 1987) and correlated to the underlying price process (Rosenberg, 1972, Black, 1975, Geske, 1979, Beckers, 1980).

In order to remedy these inaccuracies, the Black-Scholes model was generalised to allow stochastic volatility (Scott, 1987; Wiggins, 1987; Hull et al., 1987), and empirical studies show that stochastic volatility improves the performance of the models (Amin et al., 1997; Das et al., 1999; Buraschi et al., 2001).

However, introducing a second source of risk makes pricing more complicated as markets usually become incomplete and hence it is impossible to hedge a claim perfectly. Applying similar methods as Black, Scholes, and Merton did makes it possible to derive a similar PDE. Nevertheless, neither its solution nor any equivalent local martingale in this market is unique, and further assumptions or preferences must be made to obtain a unique price. Several suggestions were made, for example, minimising the risk of the corresponding hedging strategy, pricing via a super-hedging strategy and many more. (A general overview of these methods can be found in Pham, 2000. Its application to stochastic volatility models, for example in Laurent et al., 1999; Biagini et al., 2000; Heath et al., 2001; Pham, 2001; Grandits et al., 2002).

Another approach is to maximise an investor's expected utility by assuming an underlying utility function based on the investor's risk preference, which was introduced in Hodges, 1989 and has become quite popular and well studied, see for example M. H. Davis et al., 1993; Barles et al., 1998; Constantinides et al., 1999; Rouge et al., 2000; Constantinides et al., 2001; Becherer, 2001; Delbaen et al., 2002; Davis, 2006; Monoyios, 2009; Monoyios, 2010; Danilova et al., 2010).

A pricing equation for stochastic volatility models was first calculated by Sircar et al., 2004. Since then, many papers dealt with this problem or the corresponding problem of portfolio optimisation in this model (Kraft, 2005; Benth et al., 2005; Fouque et al., 2015; Boguslavskaya et al., 2016), and the optimisation problem of slightly generalised models such as stochastic interest rate, additional trading of zero bonds, and continuous consumptions has been studied (Li et al., 2009; Noh et al., 2011; Chang et al., 2013; W.-J. Liu et al., 2015; Kim et al., 2015). As the stochastic volatility models are complex and no simple solution can be established for the pricing and hedging of these models, many studies in this area deal more with numerical aspects than analytical and formal investigations of the model (see, for example, Carr et al., 1999; Floc'h et al., 2018).

The aim of the contribution is deriving a pricing equation for the Heston model, similar to the Black-Scholes equation, determining the optimal control, and calculating the residual risk process. Furthermore, potential shortcomings of the model when applied in practice should be investigated. We will show that, an approximated strategy with a positive probability of containing short positions is not the optimal time discrete strategy and hence not the optimal strategy that can be applied in practice. Hence, we put the model to the test, model a stock price process and apply the calculated optimal strategy and the optimal utility indifferent hedging strategy. We compare the results concerning options prices and portfolio wealth with the results we get when applying a delta hedge strategy with respect to the prices from the Black-Scholes model as well as the prices from the closed form solution of the Heston model (Heston, 1993)

1 The model

We assume a market with a stock S and a riskless bond B. The price of the stock S is modelled as a diffusion process satisfying

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma\sqrt{Y_t}\,\mathrm{d}W_t^0,\ \ \mathrm{d}Y_t = a(t, Y_t)\mathrm{d}t + \sigma\sqrt{Y_t}\,\mathrm{d}W_t^1$$

with $a(t, Y_t) = \kappa(\theta - Y_t)$ and $\mu > 0$. The processes W^0 and W^1 are standard Brownian motions with $\langle W_t^1, W_t^2\rangle = \rho t, \rho \in (-1,1)$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{0\le s\le T}, \mathbf{P})$ where $\mathcal{F}_s$ is the σ-algebra generated by W_s^0 and W_s^1.

The Bond can be traded, yielding a constant interest rate r. For simplicity we assume $r = 0$. Furthermore, let $g(S_T)$ be the payoff of a European-style claim.

Consider an agent investing in the stock and the bank account with a self-financing strategy π_t, where π_t denotes the proportion of wealth X_t the investor invests in the risky asset. That means, at any time t, the investor holds $\frac{\pi_t X_t}{S_t}$ stocks, and therefore, by the self-financing condition, the wealth process is given by $\mathrm{d}X_t = \frac{\pi_t X_t}{S_t}\,\mathrm{d}S_t$.

The Heston Model has some advantages over the Black-Scholes model, such as nonlognormal probability distribution (for example, fat tails), mean reverting volatility, leverage effect and many more, but also some disadvantages. Also, the fact that the medium and long-term maturity fits the implied volatility surface of option prices is a huge advantage. The parameters are, for example, arduous to estimate, and these estimations are crucial since the model reacts extremely sensitive to minor variations in the parameters.

Due to the two sources of risk, we cannot perfectly hedge the claim, so it is impossible to determine the price only by no-arbitrage arguments. We formalise this observation:

Proposition 1 *The market in the Heston model is incomplete.*

Note that the market becomes complete if it is possible to trade an asset with the price process Y_t. In this case, an equivalent local martingale measure is, by definition, an equivalent probability measure $\mathbf{Q}$ only if S_t and Y_t are local martingales with respect to Q, which would not be the case for the probability measures we just constructed in the proof.

Since the no-arbitrage pricing method, as it was applied in the Black-Scholes setting, cannot be applied to incomplete markets, other methods to determine the price of any claim has to be applied in the Heston model. This issue has been studied extensively, and numerous approaches exist.

One possibility is introducing another derivative in the market to complete the market and enable no-arbitrage pricing. For instance, this is the technique described in the articles by Zhu et al., 1998; Romano et al., 1997; Hobson et al., 1998; Davis, 2003. Other papers give a justification requirement for a specific selection of a martingale pricing measure. In the context of continuous-time stochastic volatility models, there are two prevalent criteria for selecting martingale pricing measures: the variance-optimal martingale measures and the minimal entropy martingale measures. There is a relationship between the variance-optimal martingale measures and the quadratic utility functions. Laurent et al., 1999; Biagini et al., 2000; Heath et al., 2001, among others, conducted substantial research on their use. The minimal entropy martingale measure may be related to the option valuation issue under an exponential utility function with constant absolute risk aversion; for instance, see Delbaen et al., 2002; Rheinländer, 2005; D. Hobson, 2004. Henderson et al., 2008 examined utility-based indifference pricing of contingent claims using stochastic volatility models. Indifference pricing derives from Hodges, 1989 and establishes a seller's/buyer's price such that the seller/buyer is indifferent to whether the claim is sold/bought. An excellent overview of the different pricing methods can be found in Henderson et al., 2008.

This paper will apply the utility indifference pricing method to the Heston model.

2 Pricing equation and hedging strategy

The investor seeks to find an optimal strategy to maximise the expected terminal utility. Therefore let U be a utility function reflecting the personal risk attitude of the investor. A popular choice as utility function is the exponential utility function $U(x) := -\exp(-\gamma x)$, $\gamma > 0$.

The advantages of this function are not only that it allows for negative wealth but also that the corresponding optimal control will be wealth-independent (see, for example, Grasselli et al., 2004). Wealth independence is crucial for obtaining a 'universal' (as opposed to individual pricing) equation, provided all investors in the market exhibit the same risk preference, which is an advantage over, for example, the power utility function. Note that in the literature, the exponential utility function is very often given as $U(x) := 1 - \frac{\exp(-\gamma x)}{\gamma}$, $\gamma > 0$, which yields the same maximising strategies.

We compare the utility of two sets of admissible strategies. The first strategy does not involve the claim at all, and the investor's goal is only to maximise the utility by maximising the expected terminal utility when trading only in the stock and the bank account. Let Φ be the set of admissible strategies, which means self-financing strategies with potentially additional requirements, such as the uniform boundedness of the wealth process from below.

We define

$$J^{(0)}(t,x,y;\pi) := \mathbf{E}[U(X_T) \mid X_t = x, Y_t = y]$$

with the value function

$$u^{(0)}(t,x,y) := \sup_{\pi\in\Phi} J^{(0)}(t,x,y;\pi),\ u^{(0)}(T,x,y) = U(x)$$

We compare these strategies with the strategies involving buying one unit of the claim. In this case, we define

$$J^{(1)}(t,x,y,s;\pi) = \mathbf{E}\big[U\big(X_T + g(S_T)\big) \mid X_t = x, Y_t = y, S_t = s\big]$$

and so the value function $u^{(0)}(t,x,s)$ is defined by

$$u^{(1)}(t,x,y,s) := \sup_{\pi\in\Phi} J^{(1)}(t,x,y,s;\pi),\ u^{(1)}(T,x,y,s) = U(x + g(y)).$$

The functions $u^{(0)}$ and $u^{(0)}$ are solutions of the Hamilton-Jacobi-Bellman equation, which has been studied extensively. A comprehensive treatment can, for example, be found in the textbook by Øksendal, 2010.

Note that J and u also depend on y, even though the payoff at time T does not. The reason for doing so is that we do not want to consider the more complex case of partial information: Since S and Y are not perfectly correlated, the σ-algebra $\mathcal{A}_t$ is larger than the σ-algebra generated by all sets $\{A \in \mathcal{A} \mid S_t = y\}$ for $\in \mathbb{R}$. And because furthermore S and Y are not independent, we have $\mathbf{E}[U(X_T) \mid X_t = x] \neq \mathbf{E}[U(X_T) \mid \mathcal{A}_t]$.

Another advantage of this approach is that it can easily be generalised to the case of claims that depend on Y_t such as volatility derivatives.

Now we can define the indifference buy price. Definition 2 The indifference-buy-price $p_t(x,s,y) = p(t,x,s,y)$ at time t is defined such that an agent with an initial random endowment is indifferent between doing nothing and buying the claim for that price. That means, for p_t, we have $u^{(0)}(t,x,y) = u^{(1)}(t, x + p(t,x,s,y), y, s)$.

In a complete market, where a geometric Brownian motion models the stock price, the utility indifference price coincides with the Black-Scholes price (see for example M. H. Davis et al., 1993).

The main result of this paper is the following.

Theorem 3 *The utility indifference price process* $p(t,y,s)$ *for the European claim* $g(S_T)$ *is determined by the PDE*

$$p_t + \left(a - \mu\rho - (1-\rho^2)\sigma^2 y v_y\right)p_y + \frac{1}{2}\sigma^2 p_{yy} y + \sigma^2 y\rho s p_{ys} = 0$$

with terminal condition $p(T,y,s) = g(s)$, *where* v *is a function which satisfies*

$$-v_t + \frac{1}{2}v_y^2(1-\rho^2)\sigma^2 y - \frac{1}{2}v_{yy}\sigma^2 y - v_y a + v_y\mu\rho - \frac{1}{2}\frac{\mu^2}{\sigma^2 y} = 0$$

and $v(T,y) = 0$. *The optimal hedging strategy is given by*

$$\pi_t X_t = \rho\frac{\partial p}{y}(t,Y_t,S_t) + \frac{\partial p}{s}(t,Y_t,S_t).$$

For an investor holding one claim and trading with the optimal strategy, the residual risk process is given by

$$\begin{aligned} dR_t = (\quad & \frac{\mu^2}{\gamma\sigma^2 Y} + \left(p_y - \frac{v_y}{\gamma}\right)\rho\mu + p_s\mu - p_t - p_s S\mu \\ & -p_y a - \frac{1}{2}p_{ss}S^2\sigma^2 Y - \frac{1}{2}p_{yy}\sigma^2 Y - p_{ys}\sigma^2\rho Y S\Big)\,dt - p_y\sigma\sqrt{Y}\,dW^1 \\ & + \left(\frac{\mu}{\gamma\sigma\sqrt{Y}} + p_y\sigma\sqrt{Y}\rho - \frac{v_y}{\gamma}\rho\sigma\sqrt{Y} + (1-S)p_s\sigma\sqrt{Y}\right)dW^0. \end{aligned}$$

The proof for this statement is given in section 6.1.

If the market is complete, which means $\rho \in \{-1,0,1\}$, the utility-indifferenceprice coincides with the no-arbitrage price. For $\rho = -1,1$ you can see that $\mathbf{E_Q}[g(S_T) \mid Y_t = y, S_t = s]$ solves the pricing equation for $\mathbf{Q}$ defined by $\mathbf{Q}$ by $\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} = \mathcal{E}\left(-\frac{\mu}{\sigma\sqrt{Y}}\right)$. - For $\rho = 0$, the optimal hedging strategy becomes just a plain delta hedge. If one defines a new probability measure $\mathbf{Q}$ the way it was done in definition 2.8 in Sircar et al., 2004, let γ tend to zero and apply Feynman-Kac, we obtain $\mathbf{E_Q}[g(S_T) \mid Y_t = y, S_t = s]$. The martingale measure is called the martin entropy martingale measure. Its importance for pricing theory is well studied. We can derive a connection between certain equivalent local martingale measures and PDEs by comparing our pricing PDE with the one in Sircar et al., 2004. The equations are quite similar, but Sircar and Zariphopoulou expressed the function v in terms of an equivalent local martingale measure.

3 Impossibility of short positions in the utility in- difference approach

As always, when dealing with trading strategies in continuous time, the question arises of whether they are applicable in practice and how they perform when simulated. In the case of an exponential utility function, this is particularly interesting as the utility function becomes very steep for large negative numbers. In particular, the marginal utility drops more than the probability density function, leading to the exclusion of short positions when applying the utility indifference approach in practice. This was already hinted by Gerer et al., 2016. We give here detailed proof in a slightly different setting. As short positions are applied in practice, one can conclude that real-world prices do not arise via the utility indifference method. As continuous trading is impossible in simulation or practice, we must restrict our trading strategies to simple processes, which are often used when defining the stochastic integral.

Definition 4 A process H is said to be simple if H has a representation

$$H_t = H_0 \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{n} H_i \mathbb{1}_{(t_i, t_{i+1}]}(t)$$

where $0 = t_1 < \cdots < t_{n+1} < \infty$ is a finite sequence of real numbers, $H_i \in \mathcal{F}_{t_i}$ with $|H_i| < \infty$ a.s., $0 \leq i \leq n$. The collection of simple processes is denoted with $\mathcal{S}$.

In continuous trading, a strategy is called admissible when its corresponding value process is bounded from below. As this excludes short positions for simple processes, we will amend this definition slightly. Definition 5 We call a trading strategy $H \in \mathcal{S}$ with step-times $t_0 < t_1 < \cdots < t_n$ simulation admissible if there exists a $B \in \mathbb{R}$ such that $H_{t_i} = 0$ for all i with $t_i > t$ for all t with $H \cdot S_t <= -B$. This means a strategy is admissible if investors stop trading entirely once their wealth process drops beneath a specific value.

Theorem 6 *Assume a trader wants to maximise the terminal wealth by investing in shares and holding one claim with a bounded payoff C. Then the optimal simulation admissible strategy does not involve any short positions in the stock.*

4 Simulation

To test the utility indifference pricing equation, we simulated a stock price process via the Euler-Maruyama method (see, for example, Kloeden et al., 2013).

We use the following set of parameters for our simulation:

$\kappa = 4.3758, \theta = 0.1505, \sigma = 0.3474, \mu = 0.0984, \rho = -0.2541, Y_0 = 0.0989, s_0 = 30,$

as these are parameters fitting to the S&P 500 according to Hirsa [36].[1]

As we are particularly interested in claims bounded from above and whose hedging strategy usually consists of holding short positions of the stock, we choose a bull spread containing a long position of a call with strike 30 and a short position of a call with strike 40.

The time interval is divided into 200 equally sized time steps, and we consider five portfolios:

d) Contains a claim priced by the Black-Scholes formula and hedged via simple delta hedging.
e) Contains a claim priced by the explicit Heston formula (Heston [35]) and hedged via simple delta hedging.
f) Contains a claim priced by our utility indifference pricing equation and hedged via simple delta hedging.
g) Contains a claim priced by our utility indifference pricing equation and hedged via the optimal hedging control.
h) Contains a claim priced by our utility indifference pricing equation and hedged via the optimal control for a portfolio consisting of a claim and shares.

Each portfolio starts with zero wealth, and all strategies are self-financing. We used the same value for μ to price the claim via the Black-Scholes formula. However, for the volatility, we used $\sigma = 0.128$ to adapt the volatility of a stock price driven by a geometric Brownian motion to our stock price process. We did 500 simulations in total. The main results are shown in the table below. The values were calculated by the terminal values of the simulations, and 10% stands for the 10 th percentile.

[1] It should be noted that several empirical studies of the Heston model exist, and the parameter estimations differ. For example, Ellersgaard et al., 2018 assume a much higher mean-reversion speed. The values estimated by J. Liu et al., 2003 are similar to ours.

Table no. 1: Terminal values of 500 simulations

	Wealth Mean	Wealth Variance	Wealth 10%	Wealth Median	Wealth 90%
Black Scholes	0.1724	0.4379	−0.6643	0.1137	1.0493
Heston Closed Form	−0.8407	1.4867	−2.1616	−1.1096	0.8922
Heston Utility Delta	−0.0668	0.1643	−0.5164	−0.1311	0.4882
Heston Hedging Control	0.4547	0.9405	−0.9401	0.6051	1.586
Heston Optimal	5.7941	39.2736	−2.1486	5.4543	13.934

Source: Results from the simulation performed in MATLAB and R

It is not surprising that the Heston Optimal strategy performs best. It is no hedging strategy; its goal is to maximise wealth instead of minimising risk. During the simulations, no short position was held in the stock (even though it got close to zero quite often). Comparing these strategies with a different set of parameters might be interesting. Here the drift of the stock is significantly positive, and the interest rate was set to zero. Hence it seems evident that a strategy comprising going short in the stock and long in the stock performs quite well. The data also shows that the variance in the optimal strategy exhibits a significant variance, and the tenth percentile is significantly lower than the one for three of the four hedging strategies. A somewhat surprising result is the bad performance of a delta hedge in the Heston model, where the price was calculated via the closed-form equation. Moreover, the performance seems even worse when considering longer maturities. The delta-hedged utility indifference claim performs worse on average than the optimal hedge. However, it is the strategy with the most negligible risk apart from the delta-hedged Black-Scholes model. Regarding the calculated option prices, it seems that the Closed Form Heston Model is not as sensitive regarding the maturity as the other two pricing models. There are no essential differences between the Black-Scholes price and the utility indifference price for our set of parameters.

We conclude that the utility indifference pricing and hedging strategies are valid methods. Nevertheless, further tests are necessary to see how these perform for a different set of parameters and to see whether the shortcomings of the Black-Scholes model can be overcome.

Conclusion

This paper examined the popular Heston model for pricing European-style derivatives via the utility indifference method. While it is possible to derive pricing equations and mathematically correct hedging strategies, we found that these strategies should be applied carefully in practice.

One reason is that the formal setting of utility indifference pricing in the Heston model does not allow for short positions in any optimal portfolio. Since short positions frequently happen in practice, this indicates that utility indifference pricing in the Heston model does not explain trading in the real world. Another reason results from the simulation we did in the second part of this paper. This simulation shows that the Heston model's optimal hedging strategy for utility indifference pricing does not perform well when applied in practice. The reasons for this can be manifold; most likely, the strategy and model are susceptible to parameter changes and approximations.

However, our simulation only used one set of parameters. More simulations and research should be performed to support or challenge our findings.

References

[1] AMIN, Kaushik I. and NG, Victor K. Inferring future volatility from the information in implied volatility in Eurodollar options: A new approach. *The Review of Financial Studies*. Vol. 10, no. 2, 1997. pp. 333-367. ISSN 0893-9454.

[2] BARLES, Guy and SONER, Halil Mete. Option pricing with transaction costs and a nonlinear Black-Scholes equation. *Finance and Stochastics.* Vol. 2, no. 4, 1998. pp. 369-397. ISSN 0949-2984.

[3] BECHERER, Dirk. *Rational Hedging and Valuation with Utility-Based Preferences.* Berlin, 2001. Dissertation. Technischen Universität Berlin. Available at: DOI: 10.14279/depositonce-311.

[4] BECKERS, Stan. The constant elasticity of variance model and its implications for option pricing. *The Journal of Finance*. Vol. 35, no. 3, 1980. pp. 661-673. ISSN 15406261.

[5] BENTH, Fred Espen and KARLSEN, Kenneth Hvistendahl. A PDE Representation of the Density of the Minimal Entropy Martingale Measure in Stochastic Volatility Markets. *Stochastics and Stochastics Reports*. Vol. 77, 2005. pp. 109-137. ISSN 10451129.

[6] BIAGINI, Francesca, GUASONI, Paolo and PRATELLI, Maurizio. Mean-Variance Hedging for Stochastic Volatility Models. *Mathematical Finance*. Vol. 10, no. 2, 2000. pp. 109-123. ISSN 0960-1627.

[7] BLACK, Fischer. Fact and Fantasy in the Use of Options. *Financial Analysts Journal.* Vol. 31, no. 4, 1975. pp. 36-41. ISSN 0015-198X.

[8] BLACK, Fischer and SCHOLES, Myron. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*. Vol. 81, no. 3, 1973. pp. 637-654. ISSN 1537534X.

[9] BLATTBERG, Robert C. and GONEDES, Nicholas J. A Comparison of the Stable and Student Distributions as Statistical Models for Stock Prices. *The Journal of Business*. Vol. 47, no. 2, 1974. pp. 244-80. ISSN 0021-9398.

[10] BOGUSLAVSKAYA, Elena and MURAVEY, Dmitry. An Explicit Solution for Optimal Investment in Heston Model. *Theory of Probability & Its Applications*. Vol. 60, no. 4, 2016. pp. 679-688. ISSN 004058X.

[11] BURASCHI, Andrea and JACKWERTH, Jens. The price of a smile: Hedging and spanning in option markets. *The Review of Financial Studies*. Vol. 14, no. 2, 2001. pp. 495-527. ISSN 0893-9454.

[12] CARR, Peter and MADAN, Dilip. Option valuation using the fast Fourier transform. *Journal of computational finance*. Vol. 2, no. 4, 1999. pp. 61-73. ISSN 1755-2850.

[13] CHANG, Hao and RONG, Xi-min. An investment and consumption problem with CIR interest rate and stochastic volatility. *Abstract and Applied Analysis*. Vol. 2013, 2013. ISSN 1085-3375.

[14] CONSTANTINIDES, George M. and ZARIPHOPOULOU, Thaleia. Bounds on prices of contingent claims in an intertemporal economy with proportional transaction costs and general preferences. *Finance and Stochastics*. Vol. 3, no. 3, 1999. pp. 345-369. ISSN 0949-2984.

[15] CONSTANTINIDES, George M. and ZARIPHOPOULOU, Thaleia. Bounds on derivative prices in an intertemporal setting with proportional transaction costs and multiple securities. *Mathematical Finance*. Vol. 11, no. 3, 2001. pp. 331-346. ISSN 0960-1627.

[16] DANILOVA, Albina, MONOYIOS, Michael and NG, Andrew. Optimal investment with inside information and parameter uncertainty. *Mathematics and Financial Economics*. Vol. 3, no. 1, 2010. pp. 13-38. ISSN 1862-9679.

[17] DAS, Sanjiv Ranjan and SUNDARAM, Rangarajan K. Of smiles and smirks: A term structure perspective. *Journal of Financial and Quantitative Analysis*. Vol. 34, no. 2, 1999. pp. 211-239. ISSN 0022-1090.

[18] DAVIS, Mark H. A. Optimal Hedging with Basis Risk. In: KABANOV, Yuri, LIPSTER, Robert, STOYANOV, Jordan (ed.) *From Stochastic Calculus to Mathematical Finance: The Shiryaev Festschrift*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2006. pp. 169-187. ISBN 978-3-540-30788-4.

[19] DAVIS, Mark HA. Valuation, hedging and investment in incomplete financial markets. In: HILL, James M. a Ross MOORE (ed.). *Applied Mathematics Entering the 21st Century: Invited Talks from the ICIAM 2003 Congress*. New York: Society for Industrial & Applied Mathematics, U.S., 2004, s. 49-70. ISBN 9780898715590.

[20] DAVIS, Mark HA, PANAS, Vassilios G. and ZARIPHOPOULOU, Thaleia. European option pricing with transaction costs. *SIAM Journal on Control and Optimization*. Vol. 31, no. 2, 1993. pp. 470-493. ISSN 0363-0129.

[21] DELBAEN, Freddy, GRANDITS, Peter, RHEINLÄNDER, Torsten, SAMPERI, Dominick, SCHWEIZER, Martin and STRICKER, Christophe. Exponential Hedging and Entropic Penalties. *Mathematical Finance*. Vol. 12, no. 2, 2002. pp. 99-123. ISSN 0960-1627.

[22] DRĂGULESCU, Adrian A. and YAKOVENKO, Victor M. Probability distribution of returns in the Heston model with stochastic volatility. *Quantitative Finance*. Vol. 2, no. 6, 2002. pp. 443-453. ISSN 1469-7688.

[23] DUAN, Jin Chuan. Conditionally fat-tailed distributions and the volatility smile in options. Rotman School of Management, University of Toronto, 1999. Working Paper.

[24] ELLERSGAARD, Simon and TEGNÉR, Martin. Stochastic volatility for utility maximizers: A martingale approach. *International Journal of Financial Engineering*. Vol. 5, no. 1, 2018. pp. 1850007. ISSN 2424-7863.

[25] FLEMING, Wendell and SONER, Mete. Controlled Markov Processes and Viscosity Solutions. New York: Springer, 2006. ISBN 9780387310718.

[26] FLOC'H, Le et al. An adaptive Filon quadrature for stochastic volatility models. *Journal of Computational Finance*. Vol. 22, no. 3, 2018. ISSN 1755-2850.

[27] FOUQUE, Jean-Pierre, SIRCAR, Ronnie and ZARIPHOPOULOU, Thaleia. Portfolio optimization and stochastic volatility asymptotics. Mathematical Finance, 2015. ISSN 0960-1627.

[28] GERER, Johannes and DORFLEITNER, Gregor. A note on utility indifference pricing. *International Journal of Theoretical and Applied Finance*. Vol. 19, no. 06, 2016. pp. 1650037. ISSN 0219-0249.

[29] GESKE, Robert. The valuation of compound options. *Journal of financial economics*. Vol. 7, no. 1, 1979. pp. 63-81. ISSN 0304-405X.

[30] GRANDITS, Peter and RHEINLÄNDER, Thorsten. On the minimal entropy martingale measure. *Annals of Probability*. Vol. 30, no. 3, 2022. pp. 1003-1038. ISSN 0091-1798.

[31] GRASSELLI, Matheus; HURD, Thomas, 2004. Indifference pricing and hedging in stochastic volatility models. *Papers math/0404447, arXiv.org*, 2004.

[32] HARRISON, Michael and PLISKA, Stanley R. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic processes and their applications*. Vol. 11, no. 3, 1981. pp. 215-260. ISSN 0304-4149.

[33] HARRISON, Michael and PLISKA, Stanley R. A stochastic calculus model of continuous trading: complete markets. *Stochastic processes and their applications*. Vol. 15, no. 3, 1983. pp. 313-316. ISSN 0304-4149.

[34] HEATH, David, PLATEN, Eckhard and SCHWEIZER, Martin. A comparison of two quadratic approaches to hedging in incomplete markets. *Mathematical Finance*. Vol. 11, no. 4, 2001. pp. 385-413. ISSN 0960-1627.

[35] HENDERSON, Vicky and HOBSON, David. Utility Indifference Pricing: An Overview. In: CARMONA, René (ed.). *Indifference Pricing: Theory and Applications*. Princeton: Princeton University Press, 2008. pp. 44-74. ISBN 9781400833115.

[36] HESTON, Steven L. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The review of financial studies*. Vol. 6, no. 2, 1993. pp. 327-343. ISSN 0893-9454.

[37] HIRSA, Ali. Computational Methods in Finance. Florida: CRC Press, 2016. ISBN 9781466576049.

[38] HOBSON, David. Stochastic Volatility Models, Correlation, and the q-Optimal Measure. *Mathematical Finance*. Vol. 14, no. 4, 2004. pp. 537-556. ISSN 0960-1627.

[39] HOBSON, David G. and ROGERS, Leonard CG. Complete models with stochastic volatility. *Mathematical Finance*. Vol. 8, no. 1, 1998. pp. 27-48. ISSN 0960-1627.

[40] HODGES, Stewart D. Optimal replication of contingent claims under transaction costs. *The Review of Futures Markets*. Vol. 8, 1989. pp. 223-238. ISSN 0898-011X.

[41] HULL, John and WHITE, Allan. The pricing of options on assets with stochastic volatilities. *The journal of finance*. Vol. 42, no. 2, 1987. pp. 281-300. ISSN 1540-6261.

[42] KIM, Mi-Hyun, KIM, Jeong-Hoon and YOON, Ji-Hun. Optimal portfolio selection under stochastic volatility and stochastic interest rates. *The Journal of the Korean Society for Industrial and Applied Mathematics.* Vol. 19, no. 4, 2015. pp. 417-428. ISSN 1226-9433.

[43] KLOEDEN, Peter E. and PLATEN, Eckhard. Numerical Solution of Stochastic Differential Equations. Berlin: Springer Berlin Heidelberg, 2013. ISBN 9783662126165.

[44] KRAFT, Holger. Optimal portfolios and Heston's stochastic volatility model: an explicit solution for power utility. *Quantitative Finance*. Vol. 5, no. 3, 2005. pp. 303313. ISSN 1469-7696.

[45] LAURENT, Jean Paul and PHAM, Huyên. Dynamic programming and meanvariance hedging. *Finance and Stochastics.* Vol. 3, no. 1, 1999. pp. 83-110. ISSN 0949 2984.

[46] LI, Jinzhu and WU, Rong. Optimal investment problem with stochastic interest rate and stochastic volatility: Maximizing a power utility. *Applied Stochastic Models in Business and Industry.* Vol. 25, no. 3, 2009. pp. 407-420. ISSN 1526-4025. Available at: DOI: 10.1002/ asmb . 759

[47] LIU, Jun and PAN, Jun. Dynamic derivative strategies. *Journal of Financial Economics*. Vol. 69, no. 3, 2003. pp. 401-430. ISSN 0304-405X.

[48] LIU, Wei-Jia, FAN, Shun-Hou and CHANG, Hao. Dynamics optimal portfolios with CIR interest rate under a Heston model. *WSEAS Transactions on Systems and Control*. Vol. 10, 2015. pp. 421-429. ISSN 1991-8763.

[49] MONOYIOS, Michael. Optimal investment and hedging under partial and inside information. In: ALBRECHER, Hansjörg; RUNGGALDIER,Wolfgang; SCHACHERMAYER, Walter (ed.) *Advanced Financial Modelling*. Berlin, New York: De Gruyter, 2009. pp. 371-410. ISBN 9783110213140.

[50] MONOYIOS, Michael. Utility-based valuation and hedging of basis risk with partial information. *Applied Mathematical Finance*. Vol. 17, no. 6, 2010. pp. 519-551. ISSN 1350-486X.

[51] NOH, Eun-Jung and KIM, Jeong-Hoon. An optimal portfolio model with stochastic volatility and stochastic interest rate. *Journal of Mathematical Analysis and Applications.* Vol. 375, no. 2, 2011. pp. 510-522. ISSN 0022-247X.

[52] ØKSENDAL, Bernt. *Stochastic Differential Equations: An Introduction with Applications (Universitext).* Berlin: Springer, 2010. ISBN 9783540047582.

[53] PHAM, Huyên. Hedging and Optimization Problems in Continuous-Time Financial Models. In: YOUNG, J.; CONT, R. (eds.). *Mathematical Finance: Theory and Practice*. 2000. Series in Contemporary Applied Mathematics.

[54] PHAM, Huyên. Mean-variance hedging for partially observed drift processes. *International Journal of Theoretical and Applied Finance*. Vol. 4, no. 02, 2001. pp. 263-284. ISSN 0219-0249.

[55] PHAM, Huyên. Smooth Solutions to Optimal Investment Models with Stochastic Volatilities and Portfolio Constraints. *Applied Mathematics & Optimization*. Vol. 46, no. 1, 2002. pp. 55-78. ISSN 1432-0606.

[56] RHEINLÄNDER, Thorsten. An entropyapproach to the Stein and Stein model with correlation. *Finance and Stochastics*. Vol. 9, no. 3, 2005. pp. 399-413. ISSN 09492984.

[57] ROGERS, Leonard Christopher Gordon. *Optimal Investment*. Berlin: Springer Berlin Heidelberg, 2013. ISBN 9783642352027.

[58] ROMANO, Marc and TOUZI, Nizar. Contingent claims and market completeness in a stochastic volatility model. *Mathematical Finance*. Vol. 7, no. 4, 1997. pp. 399-412. ISSN 0960-1627.

[59] ROSENBERG, Barr. *The behavior of random variables with nonstationary variance and the distribution of security prices.* Berkley: University of California, Institute of Business and Economic Research, 1972.

[60] ROUGE, Richard and EL KAROUI, Nicole. Pricing via utility maximization and entropy. *Mathematical Finance*. Vol. 10, no. 2, 2000. pp. 259-276. ISSN 0960-1627.

[61] RUBINSTEIN, Mark. Implied binomial trees. *The Journal of Finance*. Vol. 49, no. 3, 1994. pp. 771-818. ISSN 1540-6261.

[62] SCOTT, Louis O. Option pricing when the variance changes randomly: Theory, estimation, and an application. *Journal of Financial and Quantitative analysis*. Vol. 22, no. 04, 1987. pp. 419-438. ISSN 0022-1090.

[63] SIRCAR, Ronnie and ZARIPHOPOULOU, Thaleia. Bounds and asymptotic approximations for utility prices when volatility is random. *SIAM Journal on Control and Optimization*. Vol. 43, no. 4, 2004. pp. 1328-1353. ISSN 0363-0129.

[64] WIGGINS, James B. Option values under stochastic volatility: Theory and empirical estimates. *Journal of financial economics*. Vol. 19, no. 2, 1987. pp. 351-372. ISSN 0304-405X.

[65] ZHU, Yingzi and AVELLANEDA, Marco. A risk-neutral stochastic volatility model. *International Journal of Theoretical and Applied Finance*. Vol. 1, no. 02, 1998. pp. 289-310. ISSN 0219-0249.

Contact

Moritz Sohns, MSc
University of Finance and Administration
Estonská 500/3
101 00 Praha 10
Sohns@mokel.science

A. Appendix

Proofs of the results

In this section, the proofs for the results, discussed in the main part, are provided.

Proposition 7 The market in the Heston model is incomplete.

In the following proof, for stochastic processes H, X, we write $H \cdot X_t$ for the stochastic integral $\int_0^t H \, \mathrm{d}X_s$.

Proof: We define $W^{\perp} := W_t^1 - \rho W_t^0$. Then we have $\langle W^0, W^{\perp} \rangle = 0$. Now we define a new probability measure $\mathrm{d}\mathbf{Q}^{\lambda}$ with $\int_0^T \lambda_t \, \mathrm{d}t < \infty$ by its Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} = \mathcal{E}\left(-\frac{\mu}{\sigma\sqrt{Y}} \cdot W^0 + \lambda \cdot W_t^{\perp}\right).$$

By Girsanov's Theorem, the process

$$W_t^{\mathbf{Q}} := W^0 - \left\langle \frac{\mu}{\sigma\sqrt{Y}} \cdot W_t^0 + \lambda W_t^{\perp}, W_t^0 \right\rangle = W_t^0 - \frac{\mu}{\sigma\sqrt{Y_t}} t$$

is a Brownian Motion with respect to $\mathbf{Q}$ and hence

$$\frac{\mathrm{d}S_t}{S_t} = \mu \mathrm{d}t + \sigma\sqrt{Y_t} \, \mathrm{d}W_t^0 = \sigma\sqrt{Y_t} W_t^{\mathbf{Q}}$$

is a local Martingale for any λ_s. So there are infinite many equivalent local martingale measures for S_t, and with the Second Fundamental Theorem of Asset Pricing, which follows if via localisation of the theorems in Harrison et al., 1981; Harrison et al., 1983, we conclude that the market is incomplete.

The next part of this section provides the proofs for the important Theorem 3. We will split this theorem into smaller parts.

In order to derive a formula for the indifference price we examine $u^{(0)}$ and $u^{(1)}$.

Theorem 7 *We have*

$$u^0(t, x, y) = -exp(-\gamma x - v(t, y))$$

where v is a function that satisfies

$$-v_t + \frac{1}{2} v_y^2 (1 - \rho^2) \sigma^2 y - \frac{1}{2} v_{yy} \sigma^2 y - v_y a + v_y \mu \rho - \frac{1}{2} \frac{\mu^2}{\sigma^2 y} = 0 \tag{3}$$

and $v(T, y) = 0$. The optimal trading strategy is given by

$$\pi_t^{(0)} X_t = \frac{1}{\gamma}\left(\frac{\mu}{\sigma^2 Y_t} - \frac{\partial v}{\partial y}(t, Y_t)\rho\right).$$

To keep things short and palatable, we will neither deal with verification of our result nor with technical assumptions such as smoothness or Lipschitz conditions.

The following proof can, for example, be verified with Theorem 11.2.1 from [64], which also states the mild technical assumptions under which a solution of the partial differential equation (4) provides the optimal control. Furthermore, to improve readability, we will omit the indexes in the following calculations, and we write u_t for $\frac{\partial}{\partial t} u$, etcetera.

Proof: Throughout this proof, we write u for u^0. We apply Itô 's Lemma and obtain

$$\begin{aligned} \mathrm{d}u(t, X_t, Y_t) = \quad & u_t^0 \, \mathrm{d}t + u_x \, \mathrm{d}X + u_y \, \mathrm{d}Y + \frac{1}{2} u_{xx} \, \mathrm{d}\langle X \rangle + \frac{1}{2} u_{yy} \, \mathrm{d}\langle Y \rangle + u_{xy} \, \mathrm{d}\langle X, Y \rangle \\ = \quad & u_t^0 \, \mathrm{d}t + u_x (\pi X \mu) \mathrm{d}t + u_x \pi_t X_t \sigma \sqrt{Y_t} \, \mathrm{d}W^0 + u_y a \, \mathrm{d}t \\ & + u_y \sigma \sqrt{Y} \, \mathrm{d}W^1 + \frac{1}{2} u_{xx} (\pi X \sigma)^2 Y \, \mathrm{d}t + \frac{1}{2} u_{yy} \sigma^2 Y \, \mathrm{d}t + u_{xy} \rho \pi X \sigma^2 Y \, \mathrm{d}t \\ = \quad & \Big(u_t^0 + u_x (\pi X \mu) + u_y a + \frac{1}{2} u_{xx} (\pi X \sigma)^2 Y + \frac{1}{2} Y u_{yy} \sigma^2 \\ & + u_{xy} \rho \pi X \sigma^2 Y \Big) \mathrm{d}t + u_x \pi X \sigma \sqrt{Y} \, \mathrm{d}W^0 + u_y \sigma \sqrt{Y} \, \mathrm{d}W^1. \end{aligned}$$

By the Davis-Varaiya martingale principle of optimal control (see for example Theorem 1.1 in Rogers [55]), the process $u(t, X_t, Y_t)$ is a martingale for an optimal process $\hat{\pi}_t$. So, we conclude that the drift must be zero for the optimal control $\hat{\pi}$. Hence, we conclude

$$u_t^0 + u_x(\pi x \mu) + u_y a + \frac{1}{2} u_{xx} (\pi x \sigma)^2 y + \frac{1}{2} y u_{yy} \sigma^2 + u_{xy} \rho \pi y \sigma^2 = 0 \tag{4}$$

In order to find the optimal control, we differentiate with respect to π and obtain

$$u_x x \mu + u_{xx} \pi x^2 \sigma^2 y + u_{xy} \rho x y \sigma^2 = 0.$$

Hence, we have

$$\hat{\pi} x = -\frac{u_x \mu - u_{xy} \rho y \sigma^2}{u_{xx} \sigma^2 y} = -\frac{u_x \mu}{u_{xx} \sigma^2 y} - \frac{u_{xy} \rho}{u_{xx}}. \tag{5}$$

Putting this into equation (4), we get

$$0 = u_t + \frac{1}{2} u_{yy} \sigma^2 Y - \frac{1}{2} \frac{u_{xy}^2 \rho^2 \sigma^2 Y}{u_{xx}} + a u_y - \frac{u_x u_{xy} \mu \rho}{u_{xx}} - \frac{1}{2} \frac{u_x^2 \mu^2}{u_{xx} \sigma^2 Y}. \tag{6}$$

This is a non-linear PDE, and there is no straightforward way to solve it. In Sircar et al., 2004, T. Zariphopoulou introduced a certain power transformation, a so-called 'distortion power' and obtained a linear PDE. We try a different ansatz here, which goes back to Fleming et al., 2006 and was applied in similar settings for example in Pham, 2002 and Benth et al., 2005. We set

$$u(t,x,y) = -\exp(-\gamma x - v(t,y)). \tag{7}$$

Now we get for:

$$u_t = -uv_t,\ u_x = -\gamma u,\ u_{xx} = \gamma^2 u,\ u_y = -v_y u,\ u_{yy} = uv_y^2 - uv_{yy},\ u_{xy} = \gamma v_y u$$

So, we obtain for (6)

$$0 = -v_t + \frac{1}{2}v_y^2(1-\rho^2)\sigma^2 y - \frac{1}{2}v_{yy}\sigma^2 y - v_y a + v_y \mu\rho - \frac{1}{2}\frac{\mu^2}{\sigma^2 y}$$

with $v(T,y) = 0$

Putting the partial derivatives into (5) yields the optimal strategy. □

Now we have a closer look at $u^{(1)}$.

Theorem 8 *We have*

$$u^{(1)}(t,x,y,s) = -exp(-\gamma x + \gamma f(t,y,s) - v(t,y))$$

where f is determined by the PDE

$$\begin{aligned} 0 = f_t \quad &+ \left(a - \mu\rho - (1-\rho^2)\sigma^2 y v_y\right) f_y + \frac{1}{2}\sigma^2 f_{yy} y + \sigma^2 y \rho s f_{ys} \\ &+ \frac{1}{2}\sigma^2 y s^2 f_{ss} + \frac{1}{2}\sigma^2\gamma(1-\rho^2) Y f_y^2 \end{aligned} \tag{8}$$

with $f(T,y,s) = g(s)$ and $v(t,y)$ is the function from Theorem 7. The optimal trading strategy is given by

$$\pi_t X_t = \frac{\mu}{\gamma\sigma^2 Y_t} + \left(\frac{\partial f}{\partial y}(t,Y_t,S_t) - \frac{1}{\gamma}\frac{\partial v}{\partial y}(t,Y_t)\right)\rho + \frac{\partial f}{\partial s}(t,Y_t,S_t).$$

Proof: We write again u for $u^{(1)}$ and omit indexes. By applying Itô, we get

$$\mathrm{d}u(t,S,X,Y) = \Big(u_t + u_x\pi X\mu + \frac{1}{2}u_{xx}\pi^2X^2\sigma^2Y + u_sS\mu + u_ya + \frac{1}{2}u_{ss}S^2\sigma^2Y + u_{xy}\rho\pi X\sigma^2Y + \frac{1}{2}u_{yy}\sigma^2Y + u_{xs}S\sigma^2Y\pi X + u_{sy}\rho\sigma^2YS\Big)\,\mathrm{d}t + M \tag{9}$$

with M being a local martingale.

So, for the optimal π, we have

$$0 = u_xx\mu + u_{xx}\pi x^2\sigma^2y + u_{xy}\rho X\sigma^2y + u_{xs}s\sigma^2yx,$$

and we obtain

$$\hat{\pi}^{(1)}x = -\frac{u_x\mu + u_{xy}\sigma^2y\rho + u_{xs}s\sigma^2y}{u_{xx}\sigma^2y} \tag{10}$$

By plugging this into equation (9) we get

$$0 = u_t - \frac{1}{2}\frac{u_x^2\mu^2}{u_{xx}\sigma^2y} + au_y + \mu su_s + \frac{1}{2}u_{yy}\sigma^2y + \frac{1}{2}u_{ss}\sigma^2ys^2 - \frac{1}{2}\frac{u_{xy}^2\sigma^2\rho^2y}{u_{xx}} - \frac{1}{2}\frac{u_{xs}^2\sigma^2ys^2}{u_{xx}} - \frac{u_xu_{xy}\mu\rho}{u_{xx}} - \frac{u_xu_{xs}\mu s}{u_{xx}} - \frac{u_{xy}u_{xs}\sigma^2\rho sy}{u_{xx}} + u_{ys}\sigma^2\rho sy. \tag{11}$$

We make a similar ansatz as before and assume

$$u(t,x,y,s) = -\exp(-\gamma x + \gamma f(t,y,s) - v(t,y)) \tag{12}$$

where v is the function from (3) and f is a function which is, so far, only defined by the PDE. We calculate the partial derivatives for:

$$\begin{aligned}
u_t &= u(\gamma f_t - v_t),\ u_x = -\gamma u,\ u_{xx} = \gamma^2u \\
u_y &= u(\gamma f_y - v_y),\ u_{yy} = u(\gamma f_y - v_y)^2 + u(\gamma f_{yy} - v_{yy}) \\
u_s &= u\gamma f_s,\ u_{ss} = u\gamma^2f_s^2 + u\gamma f_{ss},\ u_{xy} = -\gamma u(\gamma f_y - v_y) \\
u_{xs} &= -\gamma^2uf_s,\ u_{ys} = u\gamma(\gamma f_yf_s - v_yf_s + f_{ys}),
\end{aligned}$$

and by filling them into the partial derivatives in equation (11), we obtain after some reordering

$$0 = u\gamma \Big(f_t + \frac{1}{2}\sigma^2 y s^2 f_{ss} + \frac{1}{2} y\sigma^2 f_{yy} + \sigma^2\rho y s f_{ys} + a f_y + \frac{1}{2}\sigma^2\gamma(1-\rho^2) y f_y^2 \\ -\mu\rho f_y - \sigma^2(1-\rho^2) v_y y f_y\Big) \\ +u\underbrace{\left(-v_t + \frac{1}{2}(1-\rho^2)\sigma^2 v_y^2 y - \frac{1}{2}\sigma^2 v_{yy} y - a v_y + \mu\rho v_y - \frac{1}{2}\frac{\mu^2}{\sigma^2 y}\right)}_{*}.$$

Note that the term $(*)$ is equal to zero according to Theorem 7. So, since $u\gamma \neq 0$, we obtain

$$\begin{aligned} 0 = f_t + & \frac{1}{2}\sigma^2 y s^2 f_{ss} + \frac{1}{2} y\sigma^2 f_{yy} + \sigma^2\rho y s f_{ys} + a f_y + \frac{1}{2}\sigma^2\gamma(1-\rho^2) y f_y^2 \\ & -\mu\rho f_y - \sigma^2(1-\rho^2) v_y y f_y \\ = f_t + & \left(a - \mu\rho - (1-\rho^2)\sigma^2 y v_y\right) f_y + \frac{1}{2}\sigma^2 f_{yy} y + \sigma^2 y\rho s f_{ys} \\ & + \frac{1}{2}\sigma^2 y s^2 f_{ss} + \frac{1}{2}\sigma^2\gamma(1-\rho^2) y f_y^2, \end{aligned}$$

which proves equation (8) with the terminal value $f(T, y, s) = g(s)$.

In order to derive the formula for the optimal control, we plug the partial derivatives of u into equation (10), and thus we get

$$\begin{aligned} \pi x & = -\frac{u_x\mu + u_{xy}\sigma^2 y\rho + u_{xs} s\sigma^2 y}{u_{xx}\sigma^2 y} \\ & = -\frac{-\gamma u\mu - \gamma u(\gamma f_y - v_y)\sigma^2 y\rho - \gamma^2 u f_s s\sigma^2 y}{\gamma^2 u\sigma^2 y} \\ & = \frac{\mu + (\gamma f_y - v_y)\sigma^2 y\rho + \gamma f_s s\sigma^2 y}{\gamma\sigma^2 y} = \frac{\mu}{\gamma\sigma^2 y} + \left(f_y - \frac{v_y}{\gamma}\right)\rho + f_s s. \end{aligned}$$

Hence the result follows. □

Corollary 9 *We have $f(t, y, s) = p(t, y, s)$, and hence p is determined by the PDE*

$$p_t + \left(a - \mu\rho - (1-\rho^2)\sigma^2 y v_y\right) p_y + \frac{1}{2}\sigma^2 p_{yy} y + \sigma^2 y\rho s p_{ys} = 0$$

with terminal condition $p(T, y, s) = g(s)$, and the optimal hedging strategy is given by

$$\pi_t X_t = \rho\frac{\partial p}{y}(t, Y_t, S_t) + \frac{\partial p}{s}(t, Y_t, S_t).$$

Proof: The price p is determined by the equation

$$u^0(t, x, y) = u^{(1)}(t, x + p(t, y, s), y, s).$$

Using (7) and (12), we obtain

$$-\exp(-\gamma x - v(t,y)) = -\exp(-\gamma(x + p(t,y,s)) + \gamma f(t,y,s) - v(t,y)),$$

and we get the pricing equation.

The hedging strategy follows from $\pi_t = \pi_t^{(1)} - \pi_t^{(0)}$. □

Theorem 10 *For an investor holding one claim and trading with the optimal strategy, the residual risk process is given by*

$$\begin{aligned} \mathrm{d}R_t = \Big(& \frac{\mu^2}{\gamma\sigma^2 Y} + \left(p_y - \frac{v_y}{\gamma}\right)\rho\mu + p_s\mu - p_t - p_s S\mu \\ & - p_y a - \frac{1}{2}p_{ss}S^2\sigma^2 Y - \frac{1}{2}p_{yy}\sigma^2 Y - p_{ys}\sigma^2\rho Y S\Big)\,\mathrm{d}t - p_y\sigma\sqrt{Y}\,\mathrm{d}W^1 \\ & + \left(\frac{\mu}{\gamma\sigma\sqrt{Y}} + p_y\sigma\sqrt{Y}\rho - \frac{v_y}{\gamma}\rho\sigma\sqrt{Y} + (1-S)p_s\sigma\sqrt{Y}\right)\mathrm{d}W^0. \end{aligned}$$

Proof: By applying the Itô-Formula, we obtain

$$\begin{aligned} \mathrm{d}R = \ & \mathrm{d}X - \mathrm{d}p(t,Y,S) \\ = \ & \frac{\pi X}{S}\,\mathrm{d}S - \left(p_t\,\mathrm{d}t + p_y\,\mathrm{d}Y + p_s\,\mathrm{d}S + \frac{1}{2}p_{ss}\,\mathrm{d}\langle S\rangle + \frac{1}{2}p_{yy}\,\mathrm{d}\langle Y\rangle + p_{ys}\,\mathrm{d}\langle Y,S\rangle\right) \\ = \ & \pi X\mu\mathrm{d}t + \pi X\sigma\sqrt{Y}\,\mathrm{d}W^0 - \big(p_t\,\mathrm{d}t + p_s S\mu\mathrm{d}t + p_s S\sigma\sqrt{Y}\,\mathrm{d}W^0 + p_y a\,\mathrm{d}t \\ & + p_y\sigma\sqrt{Y}\,\mathrm{d}W^1 + \frac{1}{2}p_{ss}S^2\sigma^2 Y\,\mathrm{d}t + \frac{1}{2}p_{yy}\sigma^2 Y\,\mathrm{d}t + p_{ys}\sigma^2\rho Y S\,\mathrm{d}t\Big) \\ = \ & \left(\frac{\mu^2}{\gamma\sigma^2 Y} + \left(p_y - \frac{v_y}{\gamma}\right)\rho\mu + p_s\mu\right)\mathrm{d}t \\ & + \left(\frac{\mu}{\gamma\sigma\sqrt{Y}} + p_y\sigma\sqrt{Y}\rho - \frac{v_y}{\gamma}\rho\sigma\sqrt{Y} + p_s S\sigma\sqrt{Y}\right)\mathrm{d}W^0 \\ & - \left(p_t + p_s S\mu + p_y a + \frac{1}{2}p_{ss}S^2\sigma^2 Y + \frac{1}{2}p_{yy}\sigma^2 Y + p_{ys}\sigma^2\rho Y S\right)\mathrm{d}t \\ & - p_s S\sigma\sqrt{Y}\,\mathrm{d}W^0 - p_y\sigma\sqrt{Y}\,\mathrm{d}W^1. \end{aligned}$$

It is also possible to use the pricing equation for further rearrangements. However, this does not simplify the equation any further. □

Theorem 12 *Assume a trader wants to maximise the terminal wealth by investing in shares and holding one claim with a bounded payoff C. Then the optimal simulation admissible strategy does not involve any short positions in the stock.*

In the following proof, for stochastic processes H, X, we write again $H \cdot X_t$ for the stochastic integral $\int_0^t H\,\mathrm{d}X_s$.

Proof: We show that no simulated admissible strategy with $\mathbf{P}(\min H_t < 0) > 0$ is an optimal control.

Let $h > 0$ be a real number with $\mathbf{P}(\min_t H_t < h) > 0$ and we define the random variable b as the unique number (possible infinity) for that we have $t_b = \min\{t_i \in [0,T]: H_{t_i} < -h\}$. Furthermore let x be a real number with $x > \frac{M}{h}$ where M is the boundary from definition 5 and $A^x := \left\{\omega \in \Omega: S_{b+1} - S_b - \frac{H \cdot S_\tau}{h} > x\right\}$. We have $\lim_{x\to\infty} \exp(\lambda x)\mathbf{P}\left(A^x \mid \mathcal{F}_{t_b}\right) = \infty$, which can be seen with the equation (44) from Drăgulescu et al., 2002 which gives an expression for the asymptotic distribution of the returns in the Heston model.

First, we note that we have $S_{b+1} - S_b \geq 0$ for all $\omega \in A^x$. Assume $S_{b+1} - S_b < 0$, then we have of all $\omega \in A^x$

$$-h(S_{b+1} - S_b) + H \cdot S_{t_b} < -hx < -M.$$

Hence, we have

$$H \cdot S_{t_b} < -h(S_{b+1} - S_b) < -M.$$

Since S is continuous, $H \cdot S$ is also continuous and there exists a $t < t_b$ with $\cdot S_t < -M$, and thus we have $H_t = 0$ for all $t \geq t_b$ because H is admissible. This contradicts the definition of t_b and hence we conclude $S_{b+1} - S_b \geq 0$.

Now we see that we have $H_t = 0$ for all $t >= t_{b+1}$ for all $\omega \in A^x$, which is a direct consequence of H being admissible:

$$\begin{aligned} H \cdot S_{b+1} &= H_b(S_{b+1} - S_b) + H \cdot S_{t_b} \leq -h(S_{b+1} - S_b) + H \cdot S_{t_b} \\ &\leq -M + H \cdot S_{t_b} - H \cdot S_{t_b} = -M. \end{aligned}$$

Hence, we derive for all $\in A^x$:

$$\begin{aligned} H \cdot S_T + C &= H_b(S_{b+1} - S_b) + C + H \cdot S_{t_b} \leq -h(S_{b+1} - S_b) + C + H \cdot S_{t_b} \\ &\leq -h(S_{b+1} - S_b) + \text{s} \;\; C + H \cdot S_{t_b} \leq -hx + \text{s} \;\; C. \end{aligned}$$

Since the function $f(x) = \max(-U(x),0) = \max(\exp(-\gamma x),0) = (U(x))^{-}$is decreasing, we have

$$\begin{aligned} \mathbf{E}\left[f(H \cdot S + C) \mid \mathcal{F}_{t_b}\right] &\geq \mathbf{E}\left[\mathbb{1}_{A^x} f(H \cdot S + C) \mid \mathcal{F}_{t_b}\right] \geq \mathbf{E}\left[\mathbb{1}_{A^x} f(-hx + \text{s} \;\; C) \mid \mathcal{F}_{t_b}\right] \\ &= f(-hx + \text{s} \;\; C)\mathbf{E}\left[\mathbb{1}_{A^x} \mid \mathcal{F}_{t_b}\right] = f(-hx + \text{s} \;\; C)\mathbf{P}\left(A^x \mid \mathcal{F}_{t_b}\right) \end{aligned}$$

Now we can complete the proof. The definition of a random variable Y is defined as $\mathbf{E}[\max(Y,0)] - \mathbf{E}[\max(-Y,0)]$ with the condition that at least one of these two expectations is finite. Thus, by showing $\mathbf{E}[\max(-U(H \cdot S_T + C), 0] = \infty$, we can conclude $\mathbf{E}[U(S \cdot X_T + C)] = -\infty$, which means that H is no optimal control.

We calculate

$$\mathbf{E}[-U(H \cdot S_T + C)] \quad = \mathbf{E}\left[\mathbf{E}\left[f(H \cdot S_T) \mid \mathcal{F}_{t_b}\right]\right] \leq \mathbf{E}\left[f(-hx + \mathrm{s}\ \ C)\mathbf{P}\left(A^x \mid \mathcal{F}_{t_b}\right)\right]$$
$$= \exp(-\gamma\,\mathrm{s}\ \ C)\mathbf{E}\left[\exp(\gamma hx)\mathbf{P}\left(A^x \mid \mathcal{F}_{t_b}\right)\right]$$

By letting x tend to infinity and interchanging integral and limit (which we justify by dominated convergence), we get the desired result.